

and sometimes has short blunt processes projecting from its outer surface. These short processes apparently correspond to the long complex projections which in *R. batis* give rise to an irregular network, and they seem to indicate that the cortical layer of *R. circularis* essentially agrees with the alveolar layer of *R. batis*, differing chiefly in the amount of complexity. Surrounding the cortex there is a thin layer of gelatinous tissue in which capillaries ramify. This tissue evidently represents the thick gelatinous cushion which lies behind the disk in *R. batis*, and fills up the alveoli.

The stem of the cup is usually, if not always, longer than the diameter of the cup. It consists of a core of altered muscular substance, which is surrounded by a thick layer of nucleated protoplasm continuous with the cortical layer of the cup, and apparently also identical with it.

The cups are arranged in oblique rows to form a long, slightly-flattened spindle, which occupies the posterior two-thirds of the tail, being in a skate measuring 27 inches from tip to tip, slightly over 8 inches in length, and nearly a quarter of an inch in width at the widest central portion, but only about 2 lines in thickness.

The posterior three-fifths of the organ lies immediately beneath the skin, and has in contact with its outer surface the nerve of the lateral line. The anterior two-fifths is surrounded by fibres of the outer caudal muscles. It is pointed out that while the organ in *R. circularis* is larger than in *R. radiata*, it is relatively very much smaller than the organ of *R. batis*.

VII. "On Æolotropic Elastic Solids." By C. CHREE, M.A., Fellow of King's College, Cambridge. Communicated by Professor J. J. THOMSON, F.R.S. Received May 1, 1888.

(Abstract.)

This paper treats of elastic solids of various non-isotropic kinds. Its object is to obtain solutions of the internal equations in ascending integral powers of the variables, and apply them to problems of a practical kind, some of them already solved, but in an entirely different way, by Saint-Venant.

On the multi-constant theory of elasticity the equations connecting the strains and stresses contain 21 constants. As shown by Saint-Venant these reduce for one-plane symmetry to 13, for three-plane symmetry to 9, and for symmetry round an axis perpendicular to a plane of symmetry to 5.

Part I of this paper deals with one-plane symmetry. A solution is obtained of the internal equations of equilibrium complete so far as

it goes. It is employed in solving the problem, already treated by Saint-Venant, of a beam, whose length is perpendicular to the plane of symmetry, held at one end, and at the other acted on by a system of forces, whose resultant consists of a single force along the axis of the beam, and of a couple about any line in the terminal section through its centroid. The cross-section may be any whatever, including the case of a hollow beam, provided it be uniform throughout. The case when the cross-section is elliptical, and the beam exposed to equilibrating torsional couples over its ends is also treated. Results are obtained confirmatory of Saint-Venant's. They are also extended to the case of a composite cylinder, formed of shells of different materials whose cross-sections are bounded by concentric similar and similarly situated ellipses, the law of variation being the same for all the elastic constants of the solution. The limiting case of a continuously varying structure is deduced.

It is found when a beam is exposed to terminal traction, whether uniform or not, that the strain consists in part of a shear in the plane of the cross-section which is proportional to the traction; and the position of the lines in the cross-section, which being originally at right angles remain so, is determined. These lines are called *principal axes of traction*. If there are in addition two planes of symmetry through the axis of the beam, these *principal axes* are the intersections of the planes of symmetry with the cross-section.

When a beam of circular section is exposed to torsion, it is proved that warping will ensue proportional to the moment of the twisting couple. Only two diameters in the cross-section, and these mutually at right angles, remain perpendicular to the axis of the beam. These are called *principal axes of torsion*. If w denote displacement parallel to the axis of the beam, and r, ϕ denote the undisturbed polar co-ordinates of a point in the cross-section, referred to its centre as origin, and one of these axes as initial line, the law of warping is given by—

$$w \propto r^2 \sin 2\phi.$$

There is in general no connexion between the positions of the principal axes of traction and of torsion, as the expressions giving their inclination to the axes of co-ordinates contain wholly different elastic constants; but for three-plane symmetry of the kind already mentioned they coincide. When the material is symmetrical round the axis of the beam, the shear and the warping of course are found to vanish. It is pointed out how by means of these various properties the nature of the material may be investigated experimentally.

Part II treats of a material symmetrical round an axis, that of z , and having the perpendicular plane one of symmetry. A general solution of the internal equations of equilibrium is obtained, sup-

posing no bodily forces to act. The solution involves arbitrary constants, and consists of a series of parts, each composed of a series of terms involving homogeneous products of the variables, such as $x^l y^m z^{n-l-m}$, where l, m, n are integers, and n is greater than 3. The case $n = 7$ is worked out numerically as an illustration. The terms involving powers of the variables, the sum of whose indices is less than 4, are then obtained by a more elementary process, and these alone are required in the applications which follow. These terms arrange themselves in groups *associated* with certain constants in the expression found for the dilatation.

The first application of the solution is to "Saint-Venant's problem" for a beam of elliptical cross-section. The problem is worked out without introducing any assumptions, and a solution obtained, which is thus directly proved to be the only solution possible if powers of the variables above the third be neglected. Certain groups of *associated constants* vanish completely, and the remaining arbitrary constants express themselves very simply in terms of the terminal forces, all the constants of one group depending on one only of the components of the system of forces.

Part III consists of an application of the second portion of the solution of Part II to the case of a spheroid, oblate or prolate, and of any eccentricity, rotating with uniform angular velocity round its axis of symmetry, oz , which is also the axis of symmetry of the material. The surface of the spheroid is supposed free of all forces.

The terms depending on two only of the groups of associated constants suffice, along with a particular solution on account of the existence of what is equivalent to the occurrence of bodily forces, to satisfy all the conditions of the problem, and the strains are determined explicitly.

The limiting form of the solution when the polar axis of the spheroid is supposed to diminish indefinitely, while the equatorial remains unchanged, is applied to the case of a thin circular disk rotating freely about a perpendicular to its plane through its centre. The solution so obtained is shown to satisfy all the conditions required for the circular disk, except that it brings in small tangential surface stresses depending on terms of the order of the thickness of the disk. According to this solution the disk increases in radius, and diminishes everywhere in thickness, especially near the axis, so as to become biconcave. All, originally plane, sections parallel to the faces become very approximately paraboloids of revolution, the latus rectum of each varying inversely as the square of the angular velocity into the original distance of the section considered from the central section.

Again, by supposing the ratio of the polar to the equatorial diameter of the spheroid to become very great, a surface is obtained which near the central plane, $z = 0$, of the spheroid differs very little

from that of a right circular cylinder. The corresponding form of the solution obtained for the spheroid, when the ratio of the polar to the equatorial diameter becomes infinite, may thus be expected to apply very approximately to the portions of a rotating cylinder not too near the ends, and thus for a long thin cylinder to be for all practical purposes satisfactory. This is verified directly, and it is shown that this solution is in all respects as approximately true as that universally accepted for Saint-Venant's problem. According to the solution the cylinder shortens, and every cross-section increases in radius but remains plane. The shortening and the increase in the radius are, of course, proportional to the square of the angular velocity.

Part IV treats of the longitudinal vibrations of a bar of uniform circular section and of material the same as in Part II. Assuming strains of the form—

$$\text{radial} = r \psi(r) \cos(pz - \alpha) \cos kt,$$

$$\text{longitudinal} = \phi(r) \sin(pz - \alpha) \cos kt,$$

$\phi(r)$ is found in terms of $\psi(r)$ by means of the equations established in Part II. From these equations is deduced a differential equation of the fourth order for $\psi(r)$, and for this a solution is obtained containing only positive integral even powers of r . A relation exists determining all the constants of the solution in terms of the coefficients a_0 and a_2 of r^0 and r^2 . In applying this solution to the problem mentioned, terms containing powers of r above the fourth are neglected, and it is shown to what extent the results obtained are approximate.

On the curved surface the two conditions that the normal and tangential stresses must vanish determine a_2 in terms of a_0 , and lead to the following relation between k and p —

$$k = p \left(\frac{M}{\rho} \right)^{\frac{1}{2}} \left\{ 1 - \frac{1}{4} p^2 a^2 \sigma^2 \right\}.$$

Here ρ denotes the density and a the radius of the beam, while M is Young's modulus, and σ the ratio of lateral contraction to longitudinal expansion for terminal traction. This agrees with a result obtained by Lord Rayleigh* on a special hypothesis.

Proceeding to the terminal conditions, it is shown how p is determined from the conditions as to the longitudinal motion at the ends being either quite free or entirely non-existent. Since a_0 depends only on the amplitude of the vibrations, we are left with no arbitrary constant undetermined. If the bar be so "fixed" at its ends, that the radial motion is unobstructed, this leads to no difficulty, but if an

* 'Theory of Sound,' vol. 1, § 157.

end be "free" a difficulty arises. At such an end the solution requires the existence of a radial stress $U \propto (2i + 1)^3 r (a^2 - r^2)/l^3$, where i is an integer depending on the number of the harmonic of the fundamental note and l denotes the length of the bar. The value given above for k thus answers to a problem differing to a certain extent from that occurring in nature in the case either of "fixed-free" or of "free-free" vibrations. There will thus be a difference in these cases between the results of experiment and those of the accepted theory, even as amended by Lord Rayleigh. This divergence will increase rapidly with the order of the harmonic, and though very small for a long thin bar will increase rapidly as the ratio of the diameter to the length is increased. Since in dealing with the conditions at the curved surface, terms of the order $(a/l)^5$ were neglected, the same remarks apply, though to a smaller extent, in the case of the "fixed-fixed" vibrations.

From the values of u and w , which are obtained explicitly, it is shown that the hypothesis made by Lord Rayleigh is true as a first, and only as a first, approximation.

The Society adjourned over the Whitsuntide Recess to Thursday, May 31st.

Presents, May 17, 1888.

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